“When Will I Ever Use This?”

“How About Today?”

Opportunities and Challenges of Modeling with Mathematics in Algebra 1

by Jason Zimba

Contents

Jason Zimba *is a longtime leader in mathematics and science education. He holds a B.A. from Williams College with a double major in mathematics and astrophysics, an M.Sc. by research in mathematics from the University of Oxford, and a Ph.D. in mathematical physics from the University of California at Berkeley. He is the author of Force and Motion: An Illustrated Guide to Newton's Laws.*

Real-World Math is Quantitative, Which Means It’s Also Algebraic

A year and a half prior to my joining Amplify, XQ Institute approached me with a request to write a piece about project-based learning in Algebra 1. I agreed because writing it would allow me to explore the potential benefits as well as the potential pitfalls of projects and modeling in Algebra 1. Most of my work on this paper took place during the height of the Covid-19 pandemic, an unprecedented upheaval for public education and a shock to society itself that revealed the stakes of math education for a modern democracy. Indeed, the onset of the pandemic made quantitative literacy newly salient in the public conversation as case counts, infection rates, and doubling times became prominent in Twitter feeds and front pages.

> **Algebra is both powerful and necessary in a world of quantities and dependencies—that is to say, our world.**

Still, even under normal conditions, daily life and society revolve around quantitative factors. As early as 2001, Robert P. Moses and Charles T. Cobb were arguing that "economic access and full citizenship depend crucially on math and science literacy," and that "math literacy—and algebra in particular—is the key to the future of disenfranchised communities."[1] That same year, Lynn Steen detailed "the rapidly increasing uses of quantitative thinking in the workplace, in education, and in nearly every other field of human endeavor."[2] Over two decades this trend has only intensified. Even in the humanities, quantitative methods are becoming increasingly visible in such forms as mathematical models for the formation of opinions in social networks[3] and experiments in philosophy.[4] Because of their value for future earnings, career prospects, life goals, and social mobility, quantitative skills are, and will remain, essential learning for every high school student.

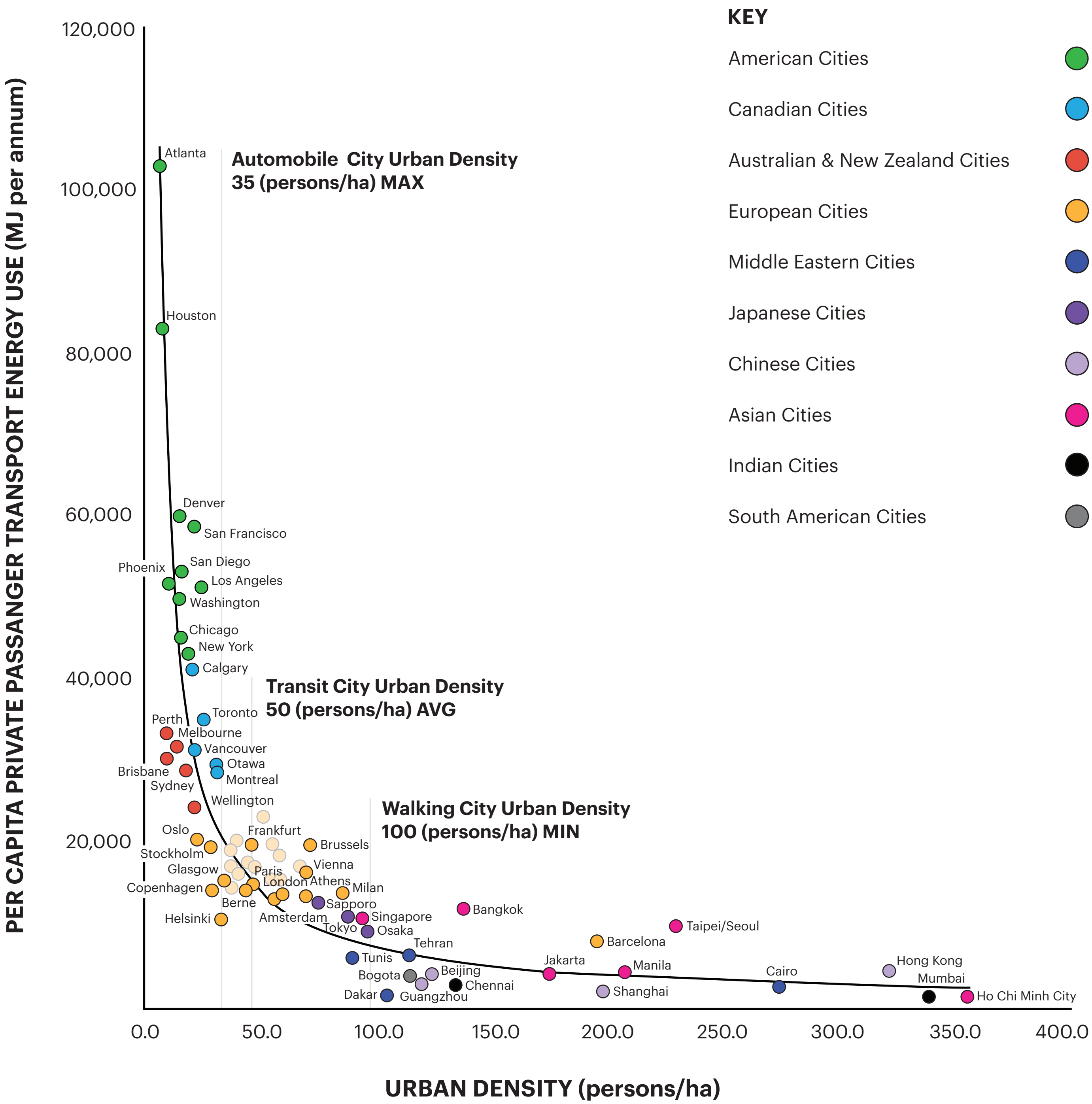
The term "quantitative" doesn't only refer to numbers. Rather, quantitative literacy draws heavily on functional thinking: a sense of how quantities depend upon one another. For example, if we suppose that some specific percentage of Covid cases will require hospitalization, then doubling the number of infections translates into doubling the number of hospitalizations as well. Or, in a financial setting, if there is a fixed fee to take out a loan, then one should be able to see that as the size of the loan increases, the fee becomes a smaller and smaller percentage of the total cost of the loan.

No numbers were used in stating the above examples; instead, words were used to name quantities that vary, and words were also used to express relationships between the quantities. Before algebra was invented, words and phrases were the only way to work with quantities and relationships. But in mathematics today, we assign variables to quantities and calculate with those variables as if they were numerical. In this way, algebra allows us to ask and answer precise questions about how the quantities could behave in different scenarios.

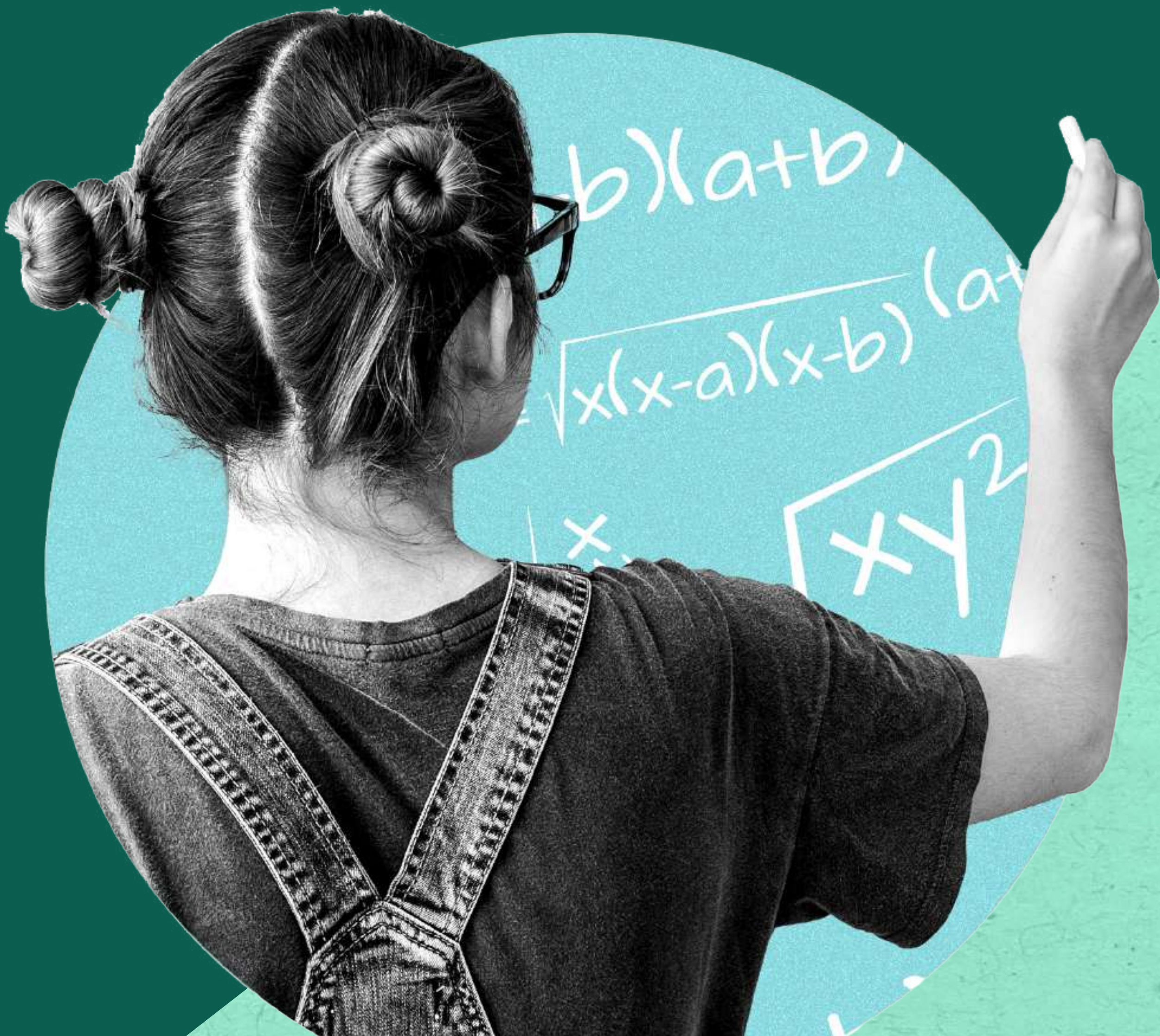
Naming variable quantities, calculating with them as if they were numbers, and investigating their dependencies, including graphically, is the art we call algebra. Algebra is both powerful and necessary in a world of quantities and dependencies—that is to say, our world.

Even statistics uses functional thinking. Nobody gathers a set of data without first conceptualizing the variables involved. A scatter plot can't even be created without having variable names for its axes. For example, in a collection of world cities, the chart shown[5] on the following page plots population density on the horizontal axis and per-capita energy use of private passenger transportation on the vertical axis. These are the two variables in play.

One thing this example shows is the relevance of rates to quantitative literacy; the plotted quantities of persons per hectare and Megajoules per person are both rates formed by division. Meanwhile, the solid curve in the plot illustrates how functional thinking helps distill the "message" of a data set: here, one qualitative message is that less dense cities tend to have more energy-intensive personal transportation. While that summary is accurate as far as it goes, as a qualitative statement it fails to capture most of the information in the graph. (That same summary would have described the data equally well had the dependence in the plot been linear.) If we can use algebra to name variables and build a function model, then we can vastly improve our ability to analyze phenomena and potentially draw actionable conclusions about them.



The Algebra of Algebra 1



The centrality of Algebra 1 to school mathematics makes Algebra 1 an important locus for efforts to improve mathematics education. In particular, a frequent call we hear is to make Algebra 1 more relevant. One source of this advice is a concern that if the curriculum doesn't engage adolescent students, then they may turn away; and when a student turns their back on mathematics, they may have closed off important life options. Before we consider 'the relevance of relevance,' however, we'll summarize the algebra of Algebra 1 itself.

Definitions of Algebra 1 have been remarkably stable over the decades. Algebra 1 is understood as a course that concentrates on linear, quadratic, and exponential functions and equations. Let's look harder at these topics and try to see the mathematics in them.

From the Topics to the Mathematics

How hard could it really be to see the mathematics in a topic? Aren't math topics already mathematics?

The distinction between topics and mathematics is admittedly a subtle one, but I think the distinction matters. When we think of the topic of exponential functions, say, the first thing that may come to mind is a certain body of characteristic problems and problem types. This is a view of topics as organizers for problems. But things go wrong when mathematics is equated to a body of problems belonging to specified categories. For example, Phil Daro concluded from the TIMSS video study[6] that, compared with teachers in high-performing countries, teachers in the U.S. spent more time demonstrating how to answer problems and less time using those problems to teach the mathematics which those problems implicate and express.[7]

It shouldn't surprise us if, after that kind of experience, the only problems that students can reliably solve are problems they've seen before (and seen recently). It's as if students have been trained to (try to) memorize something like a lookup table.[8] In other words, when a student encounters a problem, the student begins by matching the problem with the list of problem types in the left-hand column and then applies the corresponding method from the right-hand column. The fatal defect of this approach is that there are just too many problem types in school mathematics; the required lookup table is too vast for any human to remember.

Schoenfeld offers a similar analysis.[9] Recounting the first calculus course he ever taught, which began with a review of linear functions, Schoenfeld recalls: "It quickly became clear that my students thought there were many different kinds of linear functions—those described by the point-slope formula, the two-point formula, the slope-intercept formula, the

The Lookup Table, or the end-product of avoiding the mathematics

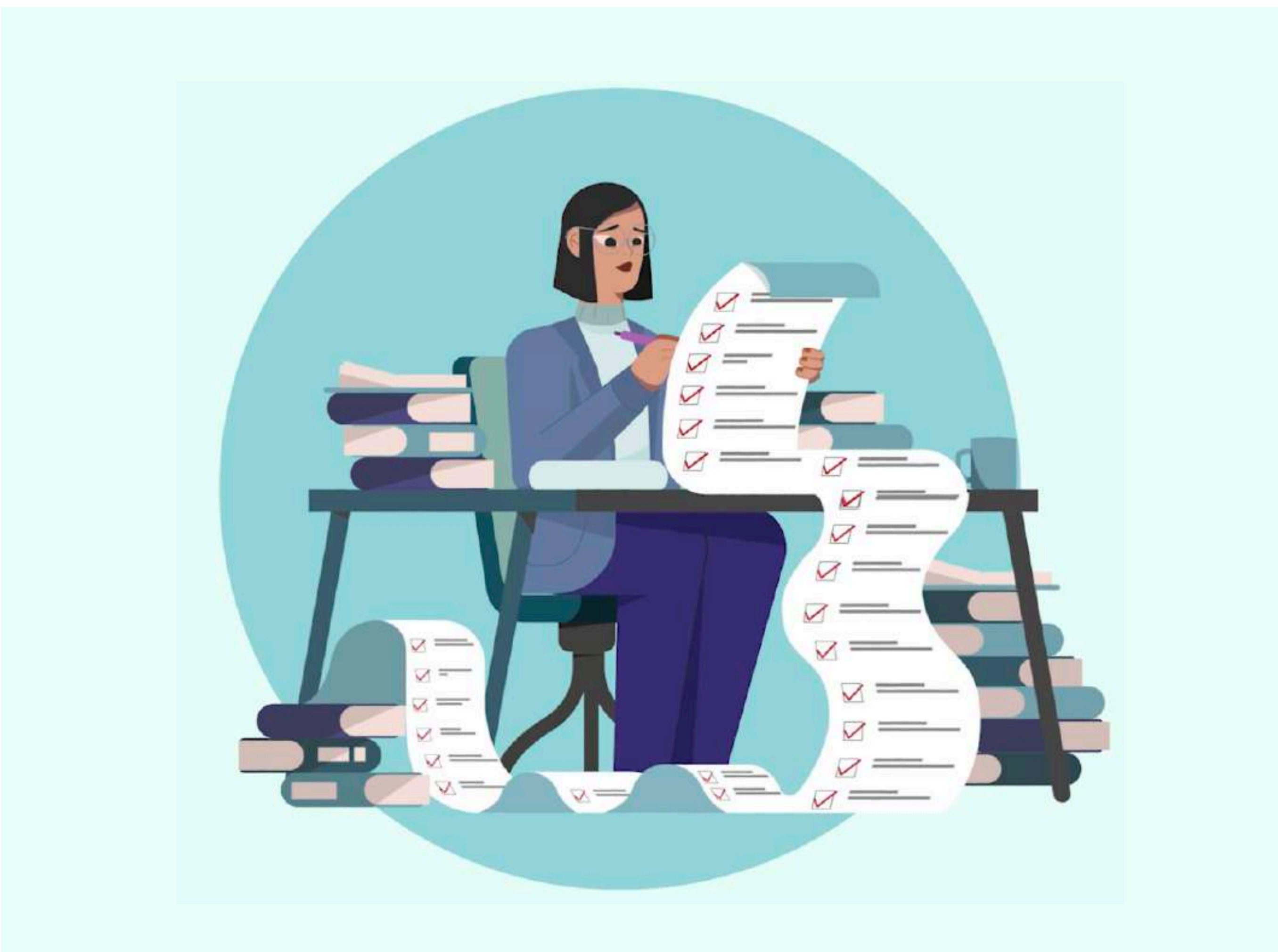
When I see this kind of problem...	...then I do this.
Problem type 1	Solution Method 1
Problem type 2	Solution Method 2
Problem type 3	Solution Method 3
Problem type 4	Solution Method 4

two-intercept formula, and those written in the general form Ax + By = C—and that each form had to be memorized and used when the teacher or task called for it (e.g., given two intercepts, you use the two-intercept formula).” Schoenfeld comments:

> An understanding at this level of grain size misses the point entirely. The main point is that: Any two distinct pieces of information about a linear function determine the function, and therefore, its equation. With that understanding, there is no need to memorize all the different forms. One can use any of the forms to obtain the equation, although some will be more convenient than others given the particular information at hand. Writing or rewriting an equation in any particular form makes it easier to ‘read’ specific information from that form.

Schoenfeld points out that “Anyone who memorizes [everything] is burdened with a huge memory load. [Without conceptual understanding], the amount to be learned can be overwhelming. The plethora of detail is one reason that performance drops off rapidly with time.”

The observations of Daro and Schoenfeld suggest that it might be valuable to have a forest-instead-of-the-trees picture of the algebra of Algebra 1. One can give a very high-level picture of school algebra by stating its three major themes: equations, functions, and generalized arithmetic (the language of symbolic expressions). At a next level of specificity (but still aiming for the forest view), the algebra of Algebra 1 might be described as shown on the following page.



Essential Algebra of Algebra 1

The function idea.[10]

- Quantities that vary and covary.
- A one-variable algebraic expression as a function specification.
- A function graph as an intelligible image of how one quantity depends on another.

Precise ideas and reliable intuitions about the behavior of a variable quantity, in prototypical cases.

- When the quantity is changing at a constant rate.
- When the quantity is changing at a constant percent rate.
- When the quantity changes quadratically.

Techniques of root finding for a one-variable equation.

- Equations as questions about a quantity; equation solving as a form of if-then argumentation.
- Exact symbolic techniques for the cases of linear and quadratic equations.
- Finding approximate roots with technology.

Techniques for simultaneously solving systems of linear two-variable equations.

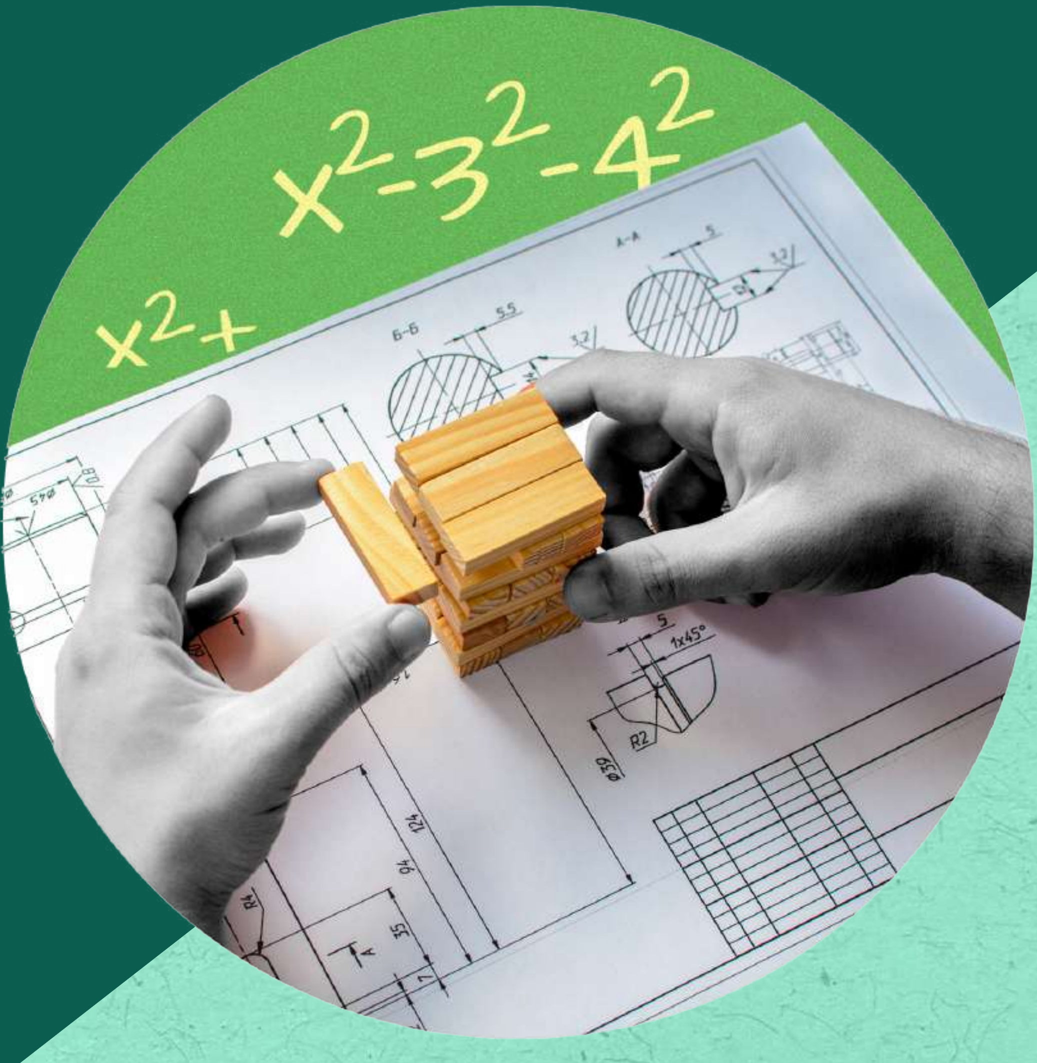
- The correspondence between a line in the coordinate plane and a two-variable equation in various equivalent forms.
- Exact symbolic techniques for solving systems.
- Finding approximate solutions to systems with technology.

Algebra as a language—a language used around the world, and throughout the world of work.

- Recognizing and naming quantities for which the value is fixed but unknown; or variable; or a matter of indifference.
- Reading algebraic expressions and interpreting them in context.
- Rewriting algebraic expressions in equivalent forms with fluency and purpose, by using properties of operations and properties of exponents.

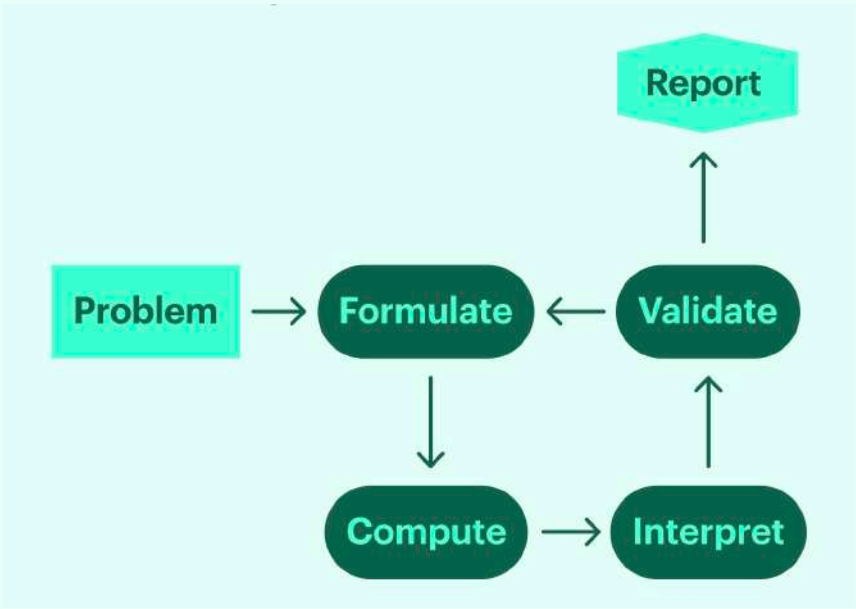
As expressed here, the essential algebra in Algebra 1 consists of a relatively small body of profound and useful knowledge—this is the “forest.”

Mathematical Modeling and Project-Based Learning



What Is Mathematical Modeling?

Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. The process of modeling mathematically is diagrammed as shown below.



Available at https://achievethecore.org/coherence-map/HS/M/more-about.

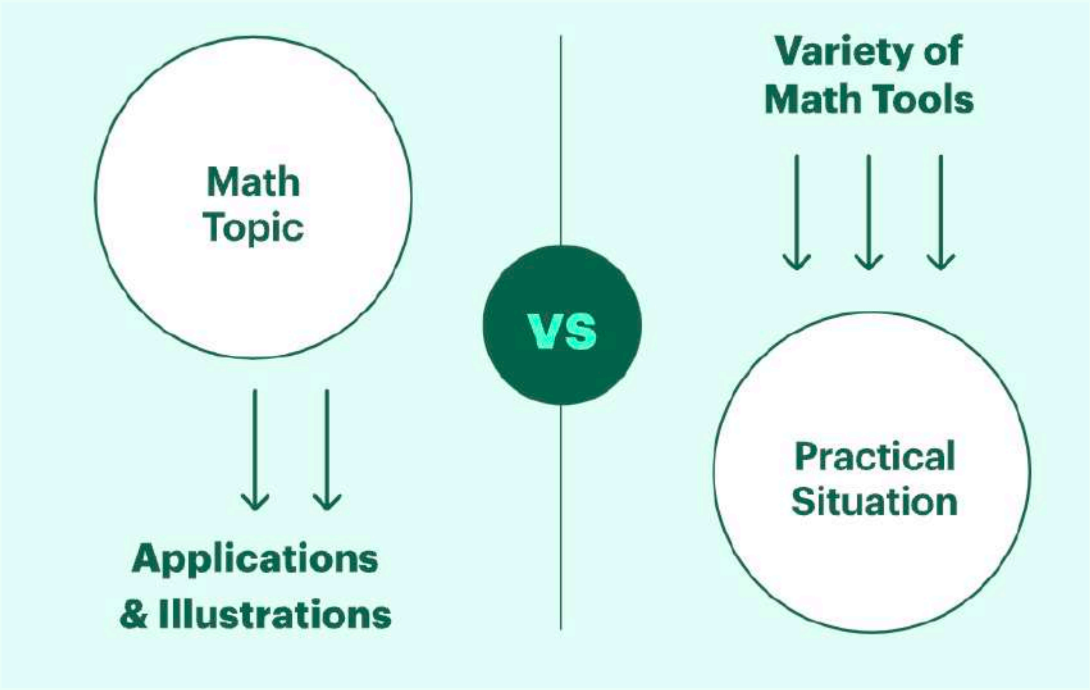
Given a problem or problem-generating situation (see the leftmost box), the student chooses relevant mathematics with which to formulate a mathematical model. For example, the heart of the model might be a function family such as exponential functions; a system of constraint equations; a probability distribution; or a simple assumption of proportionality. Because the model is mathematical, it allows computations or analyses to be performed. The results of computation have relevance to the situation which must be interpreted and checked for validity. If the results are invalid (unrealistic, contradictory, unhelpful), then the model must be adjusted or replaced—reformulated. Once valid results are obtained, a report will summarize the process and draw real-world implications.[11] Modeling tasks tend to share a family resemblance and similar features (see Some Common Features of Modeling Tasks).[12] Not every modeling task has all the features, but jointly the items listed help to flesh out mathematical modeling and evoke a zone of rich applied mathematics.

Some Common Features of Modeling Tasks

- The mathematical techniques that could be applicable in the task aren't stated explicitly, and different techniques could be valuable for analyzing the situation at hand.
    - For example, confronted with a set of bivariate data, a student might choose a function family to model the relationship between the quantities, or the student might choose to analyze marginal distributions using univariate statistics.
    - The techniques most valuable for the model might have been learned in a previous grade.
- When choosing techniques to use, the student will have to make assumptions about the situation. In complex situations, differing sets of assumptions could all be considered reasonable. Even assumptions known a priori to be implausible can sometimes lead to valuable insights.
- Independent research about needful but unknown quantities may have to be performed, and/or reasoned estimates of the quantities may have to be made (for example, by identifying the factors a quantity depends on, then making reasonable estimates of those factors).
- Quantities in the situation have meaningful units (sometimes intricate units) which are not ignored but instead are used to gain understanding of the situation.
- The task involves making decisions, making a recommendation, designing against constraints, or optimizing.
- The phenomenon or situation in the task is interesting or worthwhile beyond the academic discourse of the classroom. Modeling is an opportunity for students to apply mathematics to their community, nation, and world and to investigate and solve problems arising from their interests.
- Correspondingly, the context is not a pretext. The context is not merely a delivery mechanism for a topic.

Consider, for example, the illustration of two contrasting viewpoints on designing math applications.[13] In the viewpoint on the left, an author has a specific math topic in mind and creates a task for applications or situational illustrations of the topic. In the viewpoint on the right, the central concern is the situation, and mathematical tools are a means, not an end. Quality tasks can be built from both viewpoints. Tasks are more purely based in modeling when they are built according to the right-hand viewpoint.

Two Viewpoints on Designing Math Applications



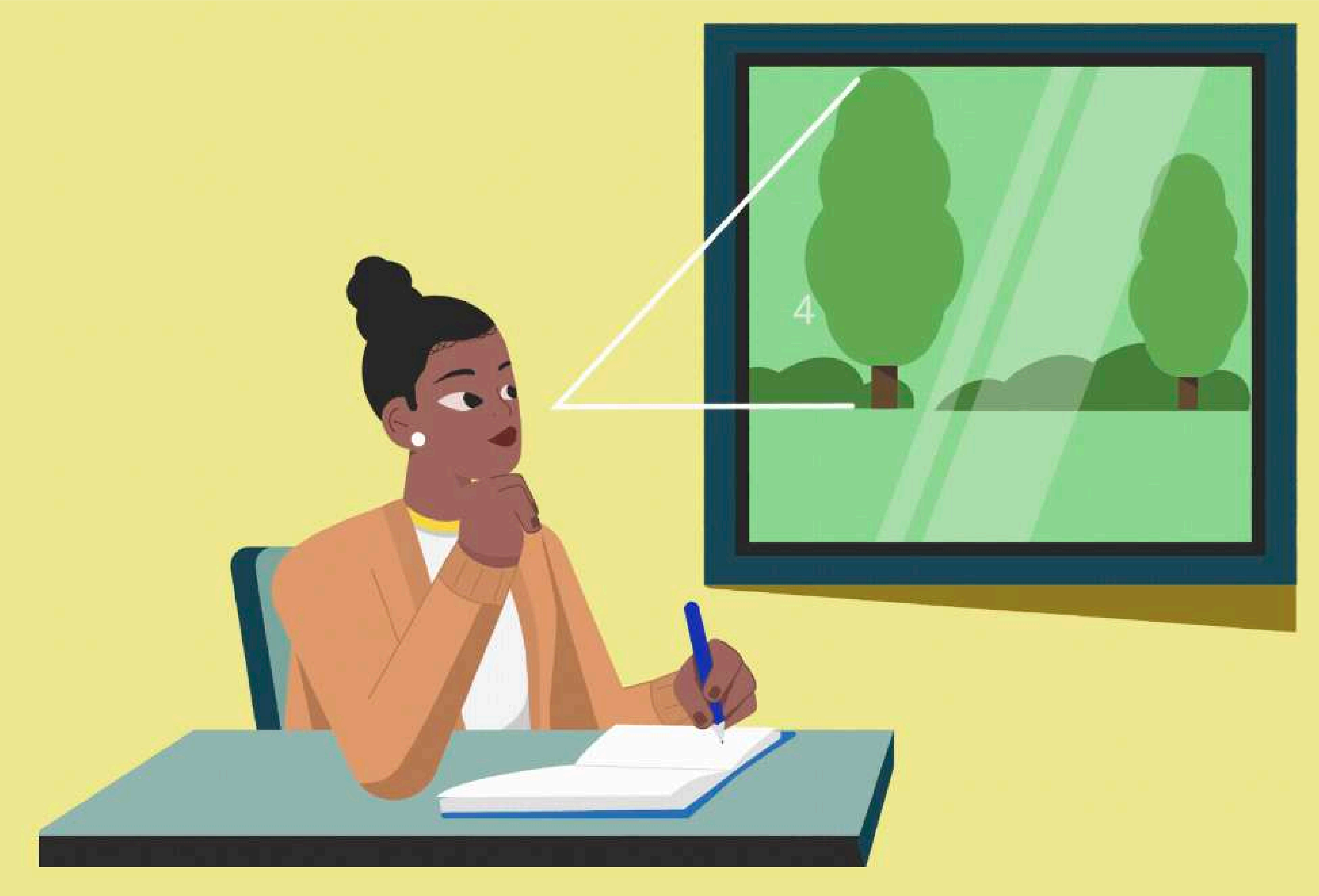
The Karnataka Task and Other Modeling Tasks

No single task could typify all dimensions of mathematical modeling, but a task that exemplifies a number of modeling features is the Karnataka task.[14] Students are presented with agricultural data from Karnataka, a state in southwest India, and challenged to develop recommendations for the use of fertilizer. Instructional materials for the task are available at https://achievethecore.org/content/upload/Modeling-Task_Karnataka.pdf.

A discussion of the modeling aspects of the Karnataka task and some possible approaches to it may be found on the digital Coherence Map at www.achievethecore.org. The Coherence Map includes modeling tasks with a range of complexity.

Other sources for modeling tasks include:

- Shell Centre for Mathematical Education
- Consortium for Mathematics and its Applications (COMAP)
- Guidelines for Assessment & Instruction in Mathematical Modeling Education (COMAP/SIAM)
- Released items from Smarter Balanced Grade 11, Claim 4 and OECD, PISA



Modeling Within Projects

One way modeling takes place in classrooms is through projects carried out by individual students or small groups of students that necessitate modeling for successful completion.[15] As Chen and Yang describe it, project-based learning "has several fundamental features or essential characteristics, as follows: inquiry, guided by the driving question, meaning that students ask their own questions, perform investigations, and develop answers; student voice and choice, meaning that students are allowed to make some decisions about the products to be constructed and how they work; revision and reflection, in which students have opportunities to use feedback to make their products better, and think about what and how they learn; and a public audience, to which students present their work."[16]

An example of such a project is Making Music from Scratch, one of a number of project-based learning prototypes being developed as part of XQ Math, a project-based learning library. The library seeks to "provide educators with resources for realizing rigorous mathematical thinking and learning through meaningful, authentic projects that students develop themselves based on a general project frame." Students undertaking the project Making Music from Scratch build a musical instrument and use it to digitally record a musical performance. See the following pages for examples of teacher and student materials from the project, as well as artifacts from students' completed projects.

By design, this project has a loosely structured frame, and students' deliverables can vary considerably, although a key condition of the project is that the instruments students build must be able to play specific notes in a song or other performance. The tuning of the instrument is performed by fitting digitally recorded sound data to a mathematical model. The value of the predictive model in the context of the project is that the model obviates the time-consuming process of trial and error, creating an authentic need for the mathematical model.

Making Music From Scratch

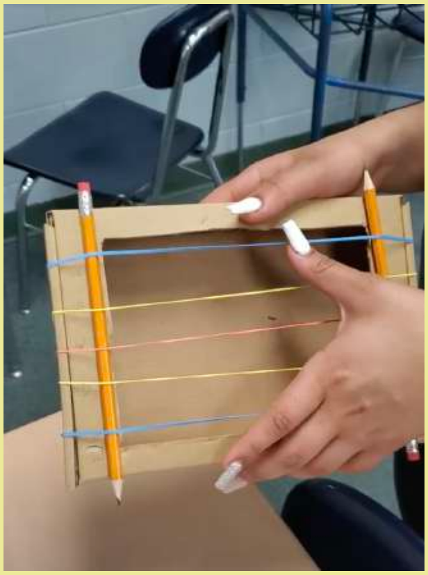
Starting with an open prompt...

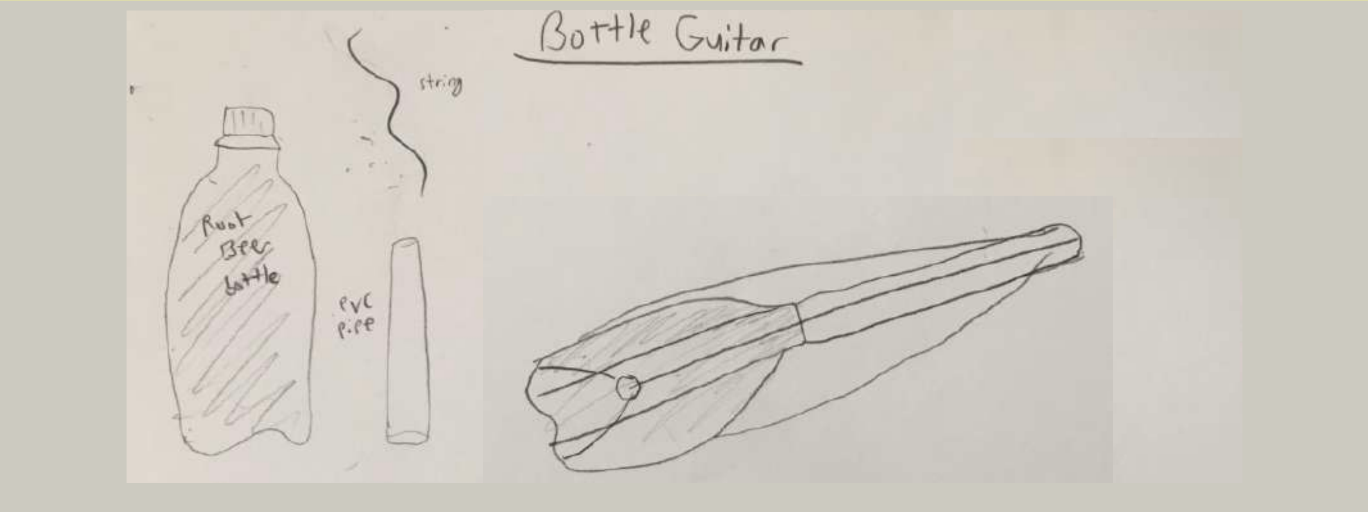
Making music you love doesn't have to require hours of lessons. In our world of mobile phones and laptops, you have a microphone in your back pocket and a recording studio in your backpack.

For this project, you're going to make an instrument from scratch to create music and share that music with others. As we dive in, you'll need to answer these three questions:

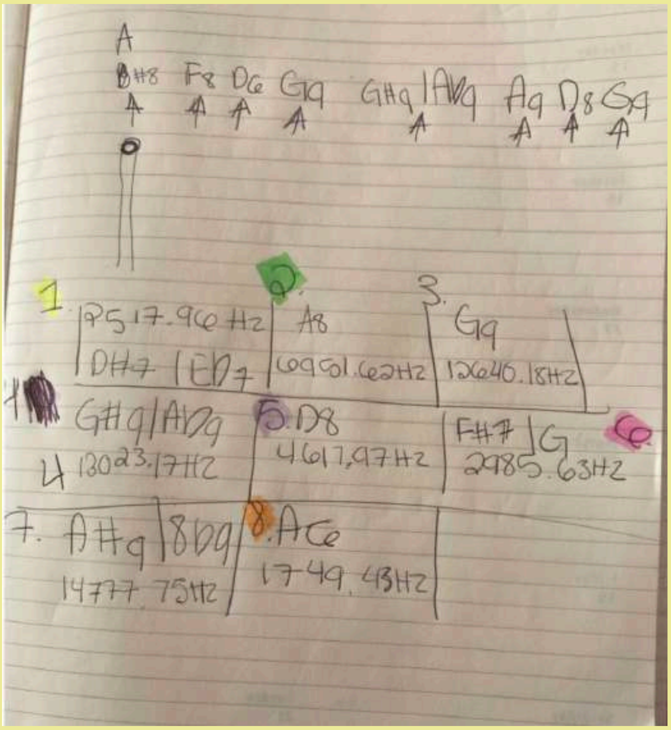
1. What music do you want to make—a song you know, or your own version of it?
2. What instrument will you create to make this music with—something with strings or holes, or something that doesn't even look like an instrument?
3. And how will you share your music—will you create an mp3, create a video of yourself playing music and share it on YouTube or TikTok, or something else?

...students build an instrument prototype.

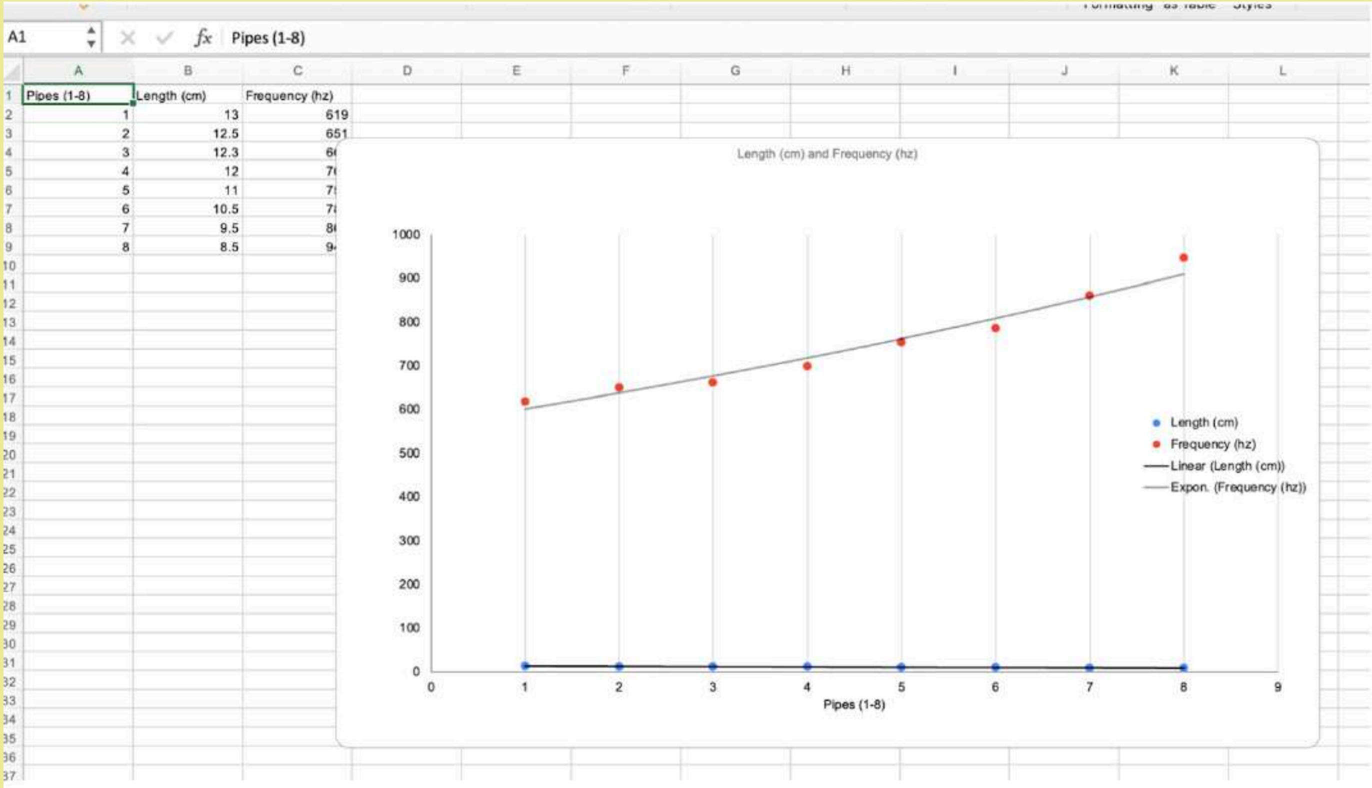




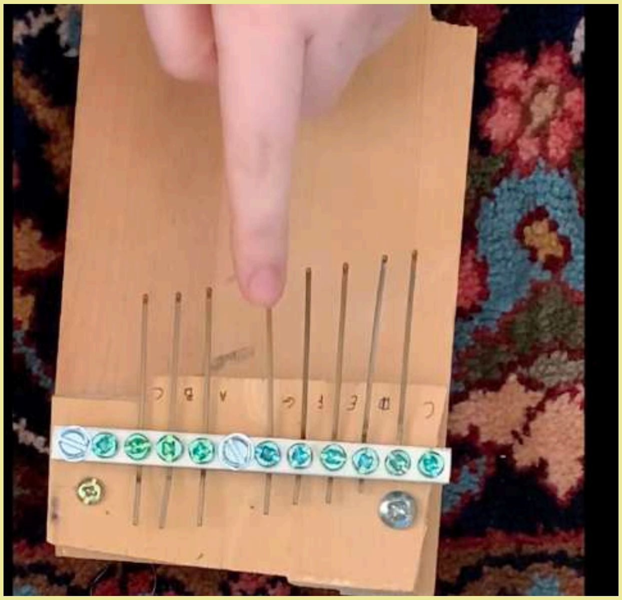
They quickly discover that they can't "tune" their instrument through trial and error, and begin collecting data...



...allowing them to generate a model of the function that governs the relationship between the frequency of notes and their instrument's variable (e.g., the length of the straws in a drinking straw pan flute)...



...allowing them to create the final version of their instrument and to play a favorite song.



Potential Benefits of Modeling and Projects

Promoting engagement

For students who are disaffected by and averse to mathematics, projects grounded in mathematical modeling could enhance motivation by prompting uses of mathematics emerging from the student's own values and interests. Holmes and Hwang describe ways in which projects can change students' relationship to mathematics for the better:

> [S]tudents began to show different characteristics of self-regulation, such as setting goals for themselves, looking for resources, seeking help, and using better learning strategies that would help them succeed. ... We gathered from indirect comments and behavior changes that students came to appreciate mathematics as a useful cognitive tool. For example, during an interview, one student shared that when she went to the store, she actually thought about unit prices and cost comparisons. She realized that her mathematics knowledge was useful. ... Initially, it seemed that many attributed their success or failure in mathematics achievement to innate ability. That could be why they said they merely wanted to pass, assuming they were not talented enough in mathematics to do better. Through PBL experiences, these students not only maintained high mathematics self-efficacy but also the belief that effort begets higher grades. [17]

Project-based work in Algebra 1 could help teachers promote such skills as setting goals, planning and organizing, persevering, and problem solving; cooperating and working on a team, resolving conflicts, and self-advocating and demonstrating agency; and coping with frustration and stress.

The format of a project can make it easier to make multiple attempts at a concept, because the issue in question remains on the table longer than a single class period. Linking perseverance in particular to the nature of the curriculum itself, Schoenfeld asks:

> Why would a student persevere? Fundamentally, perseverance is a question of agency. If a student has a sense of agency—the belief that if they work hard, they will make progress—then they are likely to try doing so. A sense of agency doesn't come from doing simple exercises that can't be considered meaningful, or from getting praise for completing such work. It doesn't come from being unsuccessful day after day at tasks that are beyond reach, or from being told what to do when you get stuck. It comes from making progress on problems that one considers meaningful challenges.[18]

Leveraging student differences

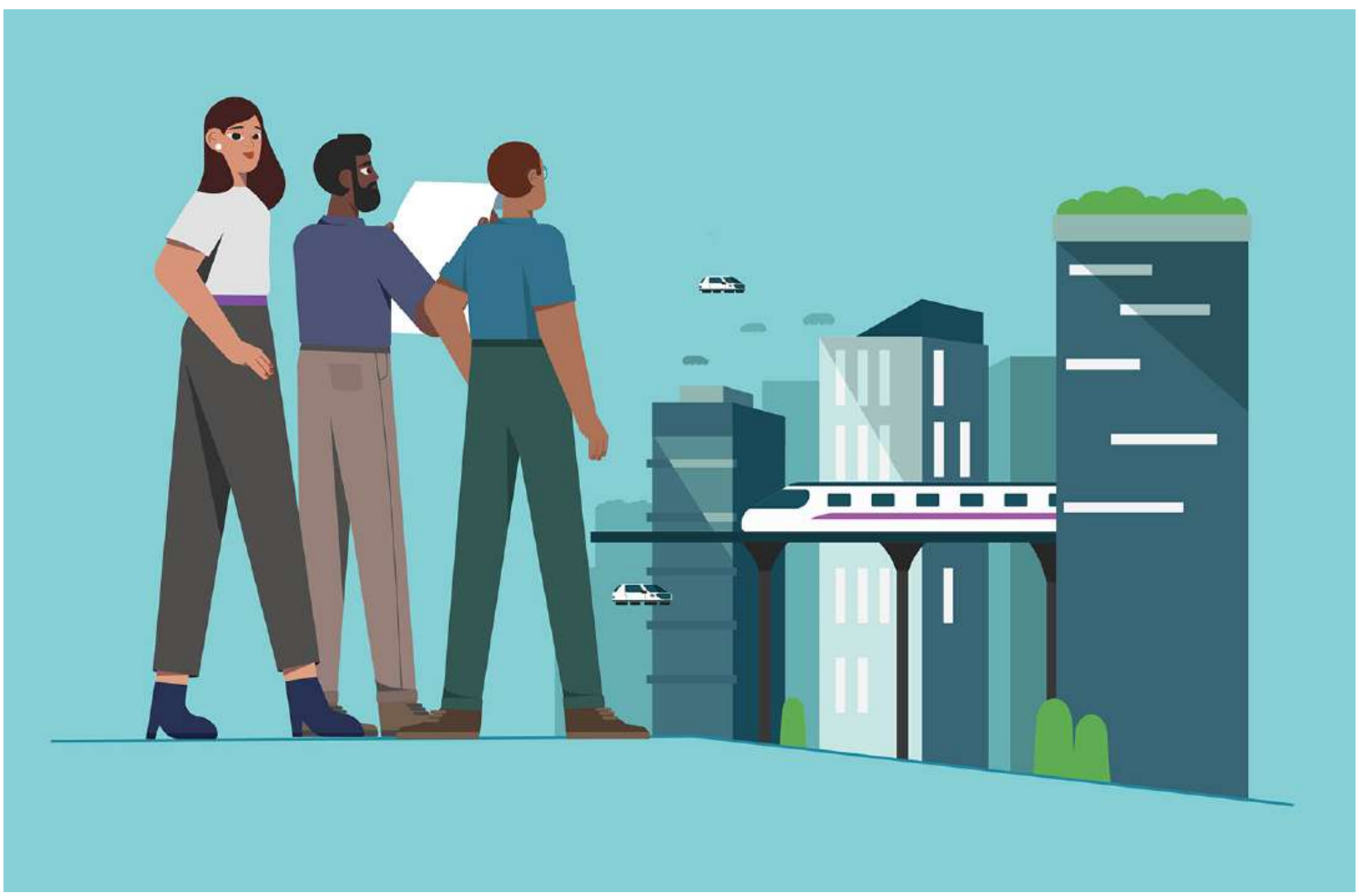
Students arrive in Algebra 1 with a wide range of knowledge of previous mathematics. Too often, schools handle this by reviewing earlier material, delaying algebra, and demotivating students with remedial assignments, inequitable teaching practices, and inequitable structures.[19] But because modeling and modeling projects, by nature, allow for a range of techniques to be applied, students doing modeling can learn from each other's mathematical choices, making differences productive. For example, two groups of students might both be working with scatterplot data—if one group uses a linear function model while the other uses a quadratic function model, the groups can discuss the advantages and disadvantages of those choices, and this conversation can invite the former group into greater comfort with the more sophisticated technique.

Building postsecondary readiness

An algebra curriculum overwhelmingly dominated by bite-sized procedural problems doesn't reflect the ways in which high school graduates will use mathematics in college, work, and life. That's because quantitative literacy in work and life is less about using advanced mathematics in scripted problems, and more about using humble mathematics in creative and sophisticated ways, as Steen explains:

> High schools focus on elementary applications of advanced mathematics whereas most people really make more use of sophisticated applications of elementary mathematics. ... Many who master high school mathematics cannot think clearly about percentages or ratios.[20]

Students will eventually enter the workplace, which is often project-based and collaborative. In addition to the workplace, many university courses require teamwork on extended projects. Projects also simulate the postgraduate world by drawing on skills such as leadership, innovation, problem sensitivity (the ability to tell when something is wrong or is likely to go wrong), originality (developing creative ways to solve a problem), multiple aspects of communication, and spatial abilities.[21] Even in advanced mathematics and science courses like calculus-based physics, students' greatest difficulties aren't usually about turning a procedural crank (say, to differentiate functions), but are more likely about being unable to use algebra in the first instance to create a model function that can be differentiated. Using algebra during high school to create mathematical models is preparation students need in order to meet future mathematical demands.



Implementing Modeling and Projects in Algebra 1

Curricular and Implementation Challenges

Ensuring that effort and activity efficiently result in mathematics learning

Project-based learning must yield mathematical learning, not just be a way to make students appear productive by creating presentations or other artifacts. Holmes and Hwang describe what can go wrong:

> In the beginning, the PBL students did not understand the mathematics behind a given project. On the surface, they could present an exemplary product (i.e., presentation). ... However, upon questioning, they did not have a rudimentary knowledge of why the algorithms were used, nor were they able to utilize common mathematics vocabulary. ... [They] presented their findings in an elaborate PowerPoint presentation. However, when asked two basic questions about how they had found the slope and what the equation represented, none of the five groups could answer either one.
>
> Although some may be concerned that students did not master the acceptable mathematics standards, these students actually took the first step in understanding that mathematics is a way of explaining the concrete world around them. As time went on, more substantive concepts were learned. Hence, while not learning grade-level mathematics content, the PBL students became more mathematics literate by using mathematics in real-life situations.[22]

But this isn't good enough. An Algebra 1 course has a mathematical job to do, and the goal of models or projects isn't to meet those requirements by redefining them away. Grading rubrics for projects should require students to articulate the mathematics used and describe its correspondences with the situation or phenomena under study; useful questions for students along these lines can be found in Guidelines for Assessment & Instruction in Mathematical Modeling Education.[23]

Covering content

During a typical project, some elements of mathematics are more likely to be activated, while other parts are less likely to be activated. For example, functions tend to play starring roles in modeling, with expressions relegated to bit parts. Therefore, modeling tasks and projects don't replace subject-matter teaching that focuses directly on the essential mathematics. Modeling doesn't obviate the need for coherent conceptual understanding of the essential mathematics; according to a National Research Council study, "Transfer is also more likely to occur when the person understands the underlying principles of what was learned."[24]

Implementing models and projects

Classroom logistics become more complex for modeling tasks that take several class periods, or for projects that take longer.

- Equity issues can arise when some students must work outside of school hours to support family members.
- Technology, in addition to its costs, requires its own forms of student and teacher expertise.
- For a teacher, helping a student get unstuck with a project is a different skill than, say, diagnosing the student's misconceptions about square roots on the evidence of written work.

The Dimension of Time

Compared to many bite-size textbook problems, modeling tasks tend to take longer to complete. And students working on projects would spend time doing assorted non-mathematical tasks required for the project—for example, doing background research. This raises the question of time, and how to spend it. There are risks on all sides here, whether by going too fast or going too slow.

Marching quickly through topics tends to leave quantitative literacy by the wayside. Quantitative literacy isn't a discrete topic, like rational exponents; it can't be covered in a single lesson or unit. Quantitative literacy is as important as any discrete topic, however, and we should learn to see a math education that doesn't build quantitative literacy as an education that's missing something. Mile-wide, inch-deep content coverage is especially unfortunate given that the essential algebra of Algebra 1 consists of a relatively short list of high-leverage ideas and fluencies (see Essential Algebra in Algebra 1). Finally, we shouldn't underestimate the importance of quantitative literacy for STEM students, who often reach university with weak conceptual fundamentals. Steen's observation that "Many who master high school mathematics cannot think clearly about percentages or ratios" calls to mind my own college physics students, who often struggled conceptually with ideas like capacitance (defined by the relation $C = Q/V$) because of difficulties with proportionality itself.[25]

Scarce time can be maximized by approaching algebra coherently, presenting it as a small body of important mathematics, rather than the currently too prevalent method of exploding the mathematics into too many topics, exploding the topics into too many problems, and exploding the problems into too many mnemonics.

Math for Life: Unlocking the Power of Algebra

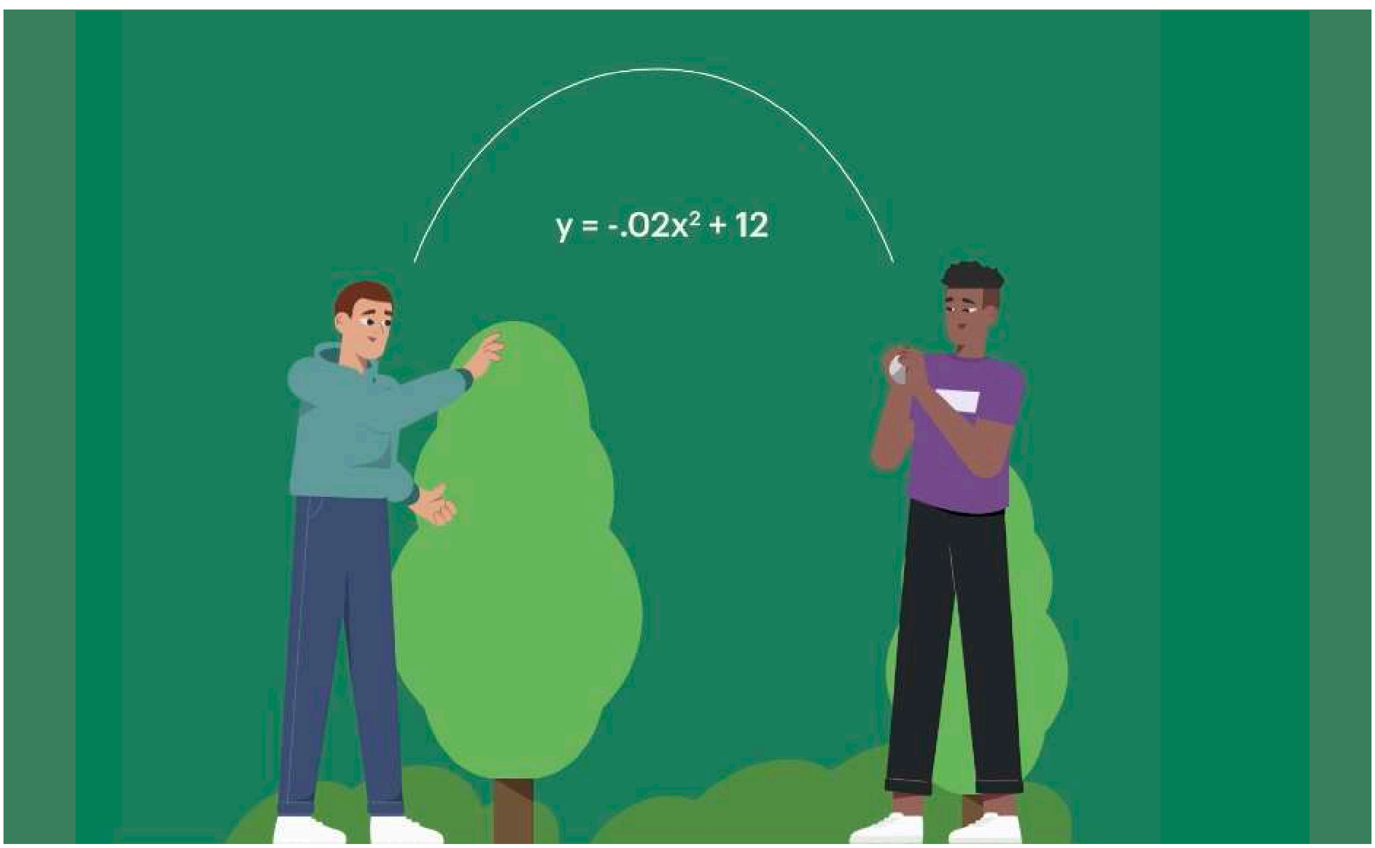
Algebra has latent power. For example, a single algebraic expression shows infinitely many calculations at once. That potential can be harnessed by entering the expression as a formula in a spreadsheet or computer program. Compared to arithmetic, algebra also expands the number of mathematical questions we can ask and answer. For example, while no finite set of cases could ever settle the question of whether a square has the greatest area among rectangles with fixed perimeter, algebra can settle that question, and optimization problems like that have many real-world applications.

Finally, algebra can be predictive of real events: if we increase the value of a temperature parameter in a formula, then we obtain a prediction of what will happen if the temperature of a physical system increases. The system under consideration might be a collapsing star, an atmosphere on a planet like Earth, or an HVAC system in an office building.

Mathematical models abstract our world's quantities and dependencies, and by doing so they help us think more clearly about situations and possibly bring events under our more effective control. In a world of quantities and dependencies, the most important quantitative skills are algebraic.

There are important reasons to increase the richness of mathematical applications in the curriculum. The ability and disposition to use mathematics in college, work, and life is important; modeling tasks are part of state standards; and implemented well, modeling and projects can offer promise for increasing engagement and improving students' postsecondary readiness.

Given the challenges, teachers who wish to move in this direction need districts to support them with professional learning that prepares them to teach and assess mathematical modeling, to facilitate students working more independently, and to use the potential of modeling to strengthen students' mathematical identity and achievement.



Notes

1. Moses, Robert P., Cobb, Charles E. (2001). Radical Equations: Civil Rights from Mississippi to the Algebra Project. United States: Beacon Press, 5.

2. Steen, L., ed. (2001). Mathematics and Democracy: The Case for Quantitative Literacy. National Council on Education and the Disciplines, 1. https://www.maa.org/sites/default/files/pdf/QL/MathAndDemocracy.pdf.

3. Del Vicario, M., Scala, A., Caldarelli, G. et al. (2017). Modeling confirmation bias and polarization. Sci Rep 7, 40391. https://doi.org/10.1038/srep40391.

4. Knobe. J., and Nichols, S. (2017). Experimental Philosophy. The Stanford Encyclopedia of Philosophy (Winter 2017 Edition), Edward N. Zalta (ed.), https://plato.stanford.edu/archives/win2017/entries/experimental-philosophy/.

5. Newman, Peter, and Kenworthy, Jeff (2021). Gasoline Consumption and Cities Revisited: What Have We Learnt? Current Urban Studies 9, 532–553.

6. Teaching Mathematics in Seven Countries: Results From the TIMSS 1999 Video Study. N.p., DIANE Publishing.

7. Daro, P. (2013). https://vimeo.com/79916037. "Correct answers are essential…but they're part of the process…. The product is the math the kids walk away with in their heads…."

8. Zimba, J. (2016). "The Life-Changing Magic of Tidying the Math Curriculum." NCTM Annual Conference Presentation, San Antonio, TX.

9. Schoenfeld, A. (2019). "Reframing teacher knowledge: a research and development agenda." ZDM, 7 May 2019.

10. For a recent analysis of the function idea in school mathematics, see: McCallum, W. (2019). Coherence and Fidelity of the Function Concept in School Mathematics. In: Weigand HG., McCallum W., Menghini M., Neubrand M., Schubring G. (eds) The Legacy of Felix Klein. ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-319-99386-7_6.

11. In common practice, tasks that are classified as modeling tasks might require only selected parts of the process (as on the PISA exam, for example). And in practice, both novices and experts might deviate from the idealized process: for example, by interpreting throughout the computation phase.

12. For a more formal list of attributes of modeling tasks, see: Zimba, J. (2013). PISA and the U.S. Common Core State Standards for Mathematics. In OECD (2013). Lessons from PISA 2012 for the United States, Strong Performers and Successful Reformers in Education. OECD Publishing. http://dx.doi.org/10.1787/9789264207585-en. For a fifteen-minute overview of mathematical modeling, see: Zimba, J. (n.d.). What is Mathematical Modeling? Fifteen-minute video at https://achievethecore.org/coherence-map/HS/M/more-about.

13. Adapted from a diagram by Malcom Swann, see: Muller, E., and Burkhardt, H. (2006). Application and Modelling for Mathematics—Overview. In W. Blum and P.L. Galbraith (Eds.), Modelling and Applications in Mathematics Education: The 14th ICMI Study, Springer, 2006. http://books.google.com/books?id=XICuY-BFaHYC.

14. Partnership for Assessment of Readiness for College and Careers Item Development ITN, Appendix F, 2012, pp. 29-30 (adapted from an original COMAP task). https://achievethecore.org/content/upload/Modeling-Task_Karnataka.pdf.

15. Science Education Resource Center (n.d.). "Model-Eliciting Activities." https://serc.carleton.edu/sp/library/mea/what.html.

16. Chen, C-H., and Yang, Y-C. (2019). Revisiting the effects of project-based learning on students' academic achievement: A meta-analysis investigating moderators. Educational Research Review 26 (2019), 71–81.

17. Holmes, V-L., and Hwang, Y. (2016). Exploring the effects of project-based learning in secondary mathematics education. The Journal of Educational Research, 109:5, 458, 459; DOI:10.1080/00220671.2014.979911.

18. Schoenfeld, 2019.

19. NCTM (2018). Catalyzing Change in High School Mathematics. Reston, VA: The National Council of Teachers of Mathematics, 15-35.

20. Steen, L. (2007). Invited testimony at the initial meeting of the Commission on Mathematics and Science Education established by the Carnegie Corporation of New York and the Institute for Advanced Study, 4. http://www.steen-frost.org/Steen/Papers/07carnegie.pdf.

21. O*Net (2020). The O*Net® Content Model. https://www.onetcenter.org/content.html.

22. Holmes and Hwang, 2016, p. 456.

23. COMAP/SIAM (2016). Guidelines for Assessment & Instruction in Mathematical Modeling Education. Sol Garfunkel and Michelle Montgomery, Eds, Table D.1, 197. http://www.siam.org/Portals/0/Publications/Reports/gaimme-full_color_for_online_viewing.pdf?ver=2018-03-19-115454-057.

24. National Research Council. 2012. Education for Life and Work: Developing Transferable Knowledge and Skills in the 21st Century. Washington, DC: The National Academies Press, 98. https://doi.org/10.17226/13398.

25. Steen, 2007, p. 4.

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